Chapter 6: Shadow Mechanics

Eclipses are shadows. A lunar eclipse is the Earth's shadow on the Moon, and a solar eclipse is the Moon's shadow on the Earth.

In order to understand the particular shadows formed during lunar and solar eclipses, let's begin by studying shadows that we are more familiar with and that we can directly measure.

Start by holding a soccer ball in the sunlight above a flat surface as shown below in Figure 6.1. Measure the diameter of the ball. Now hold the ball about five feet above the flat surface and measure the shadow of the ball. (Measure across the narrow part of the ellipse of the shadow.)

What did you find? A standard soccer ball is about 8 ¾ inches in diameter, but the darkest part of the shadow will only measure about 8 ¼″ across. The shadow is smaller than the ball.

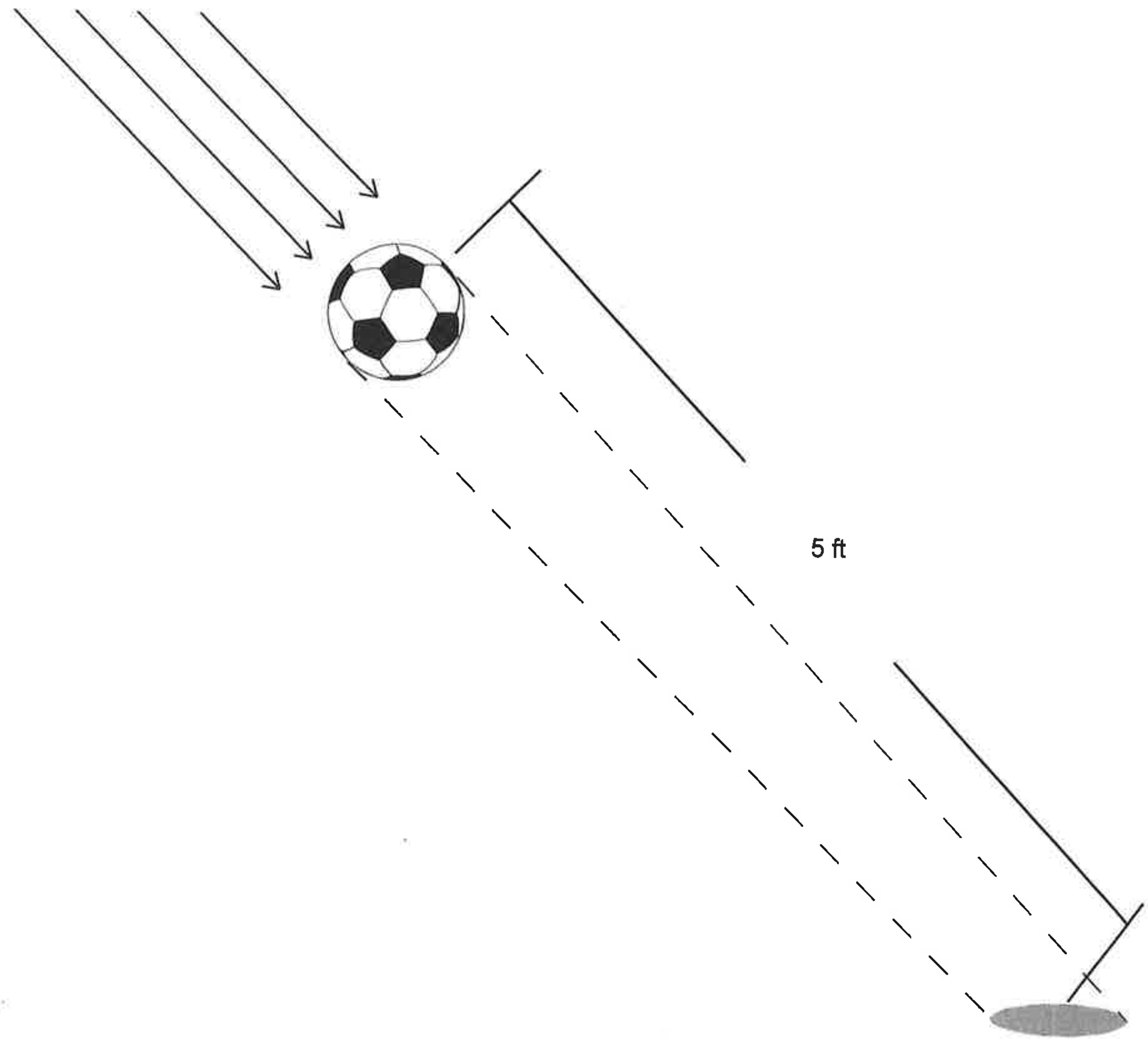


Figure 6.1: Hold a soccer ball 5′ from the surface and measure the shadow

Another thing you might notice is that the shadow is fuzzy around the edges. This fuzzy part of a shadow is called the *penumbra*, which means partial shadow. The dark middle part of the shadow is called the *umbra*, or total shadow. See Figure 6.2.

From the umbra of a shadow, no part of the source light can be seen. From the penumbral area, part of the source can be seen, but some is blocked.

We will investigate these two phenomena separately, starting with the umbra.

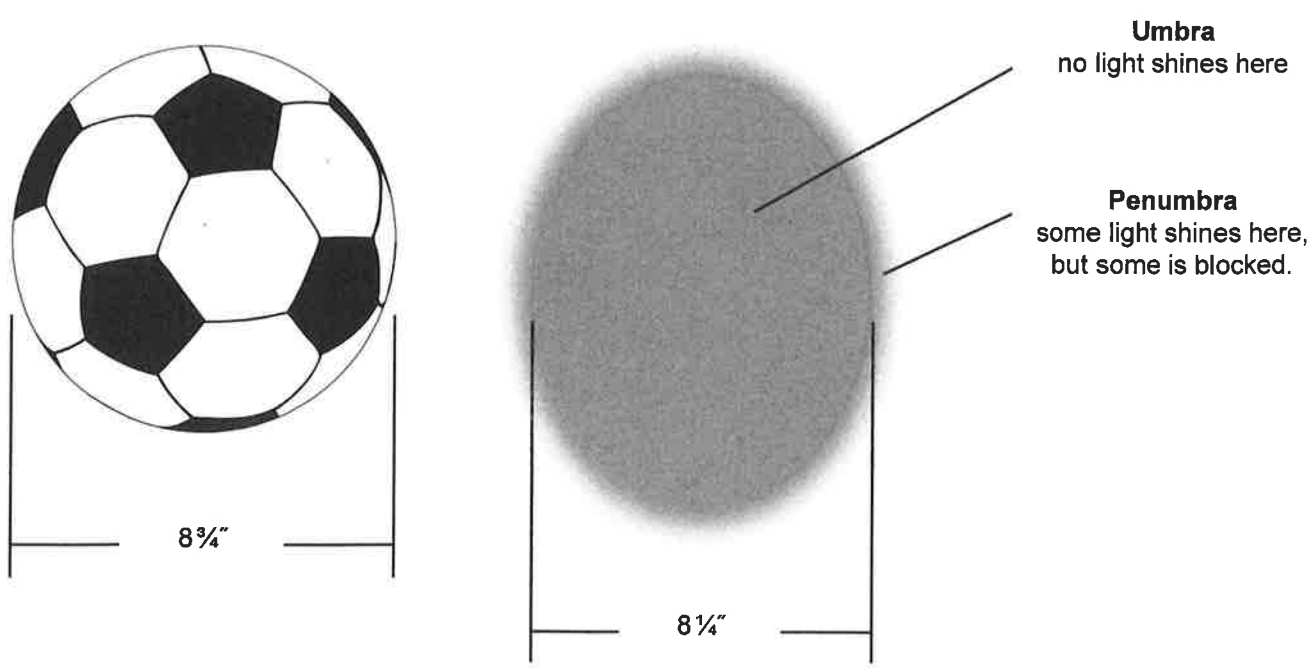


Figure 6.2: The soccer ball's shadow is smaller and fuzzy around the edges

The umbra is the dark part of the shadow. Let's think about why, when we measured it, the shadow was smaller than the ball. You might think the opposite should be true. The Sun looks small in the sky, and the soccer ball in our hand looks relatively big.

Why don't we see a large shadow like shown below in Figure 6.3?

Even though the Sun appears to be smaller than the ball, we know it is actually much larger. It just appears smaller because it is so far away.

Let's start our investigation by finding the relationship between umbra and size and distance of the light source. (We'll ignore penumbra for now and deal with it later, after we understand umbra.)

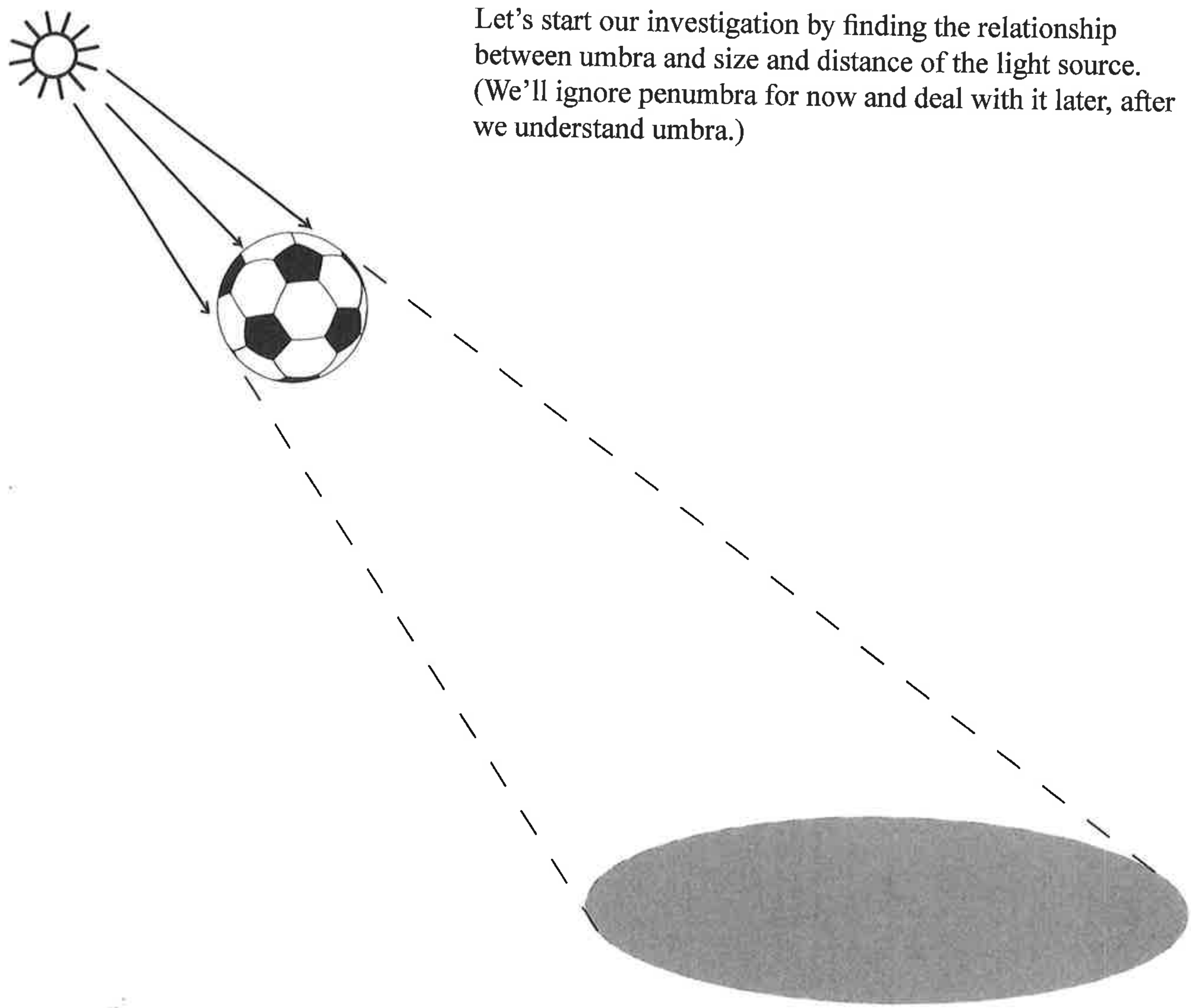


Figure 6.3: Why doesn't the Sun cast a shadow larger than the ball?

For a light source that is small and near the ball, we see a cone-shaped shadow that is larger than the ball, as shown below in Figure 6.4.

A flashlight in candle mode (with the reflector removed) works well for this demonstration. Since light travels in straight lines, any light rays aimed at the ball are blocked by the ball.

When the light source is farther away, a cone-shaped shadow is still formed, but is narrower than when the light was closer. See Figure 6.5.

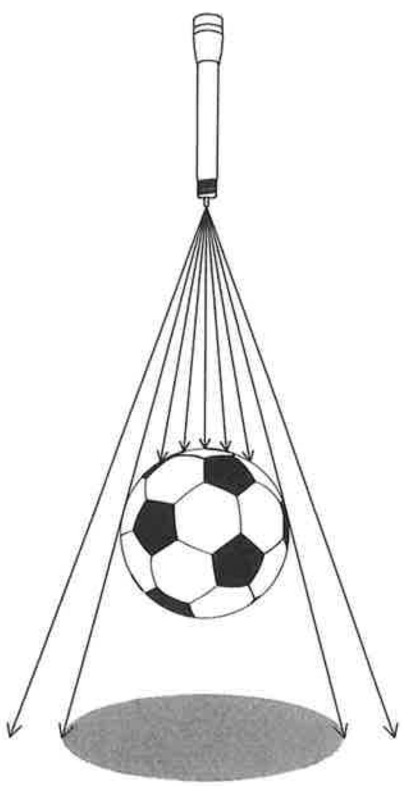


Figure 6.4: A small, close light source makes a cone-shaped shadow

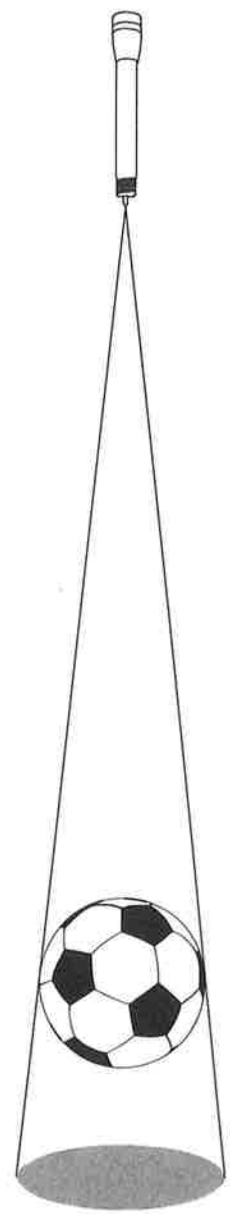


Figure 6.5: The same source farther away makes a smaller shadow

Now let's try a larger light source. A standard light bulb is about 2 3/8 inches in diameter. At the same distance, it creates a narrower shadow cone than the point source does.

A source larger than the ball, as the far right in Figure 6.6, creates a shadow cone that tapers in and is smaller than the ball. (An array of bulbs can be used to simulate a larger source.)

When the sources are moved farther from the balls, the angles become smaller. Notice that when the light source is smaller than the ball, the cone tapers out, but when the source is larger than the ball the shadow cone tapers in. See Figure 6.7.

What is the relationship between the angle of the shadow cone and size and distance of the light source?

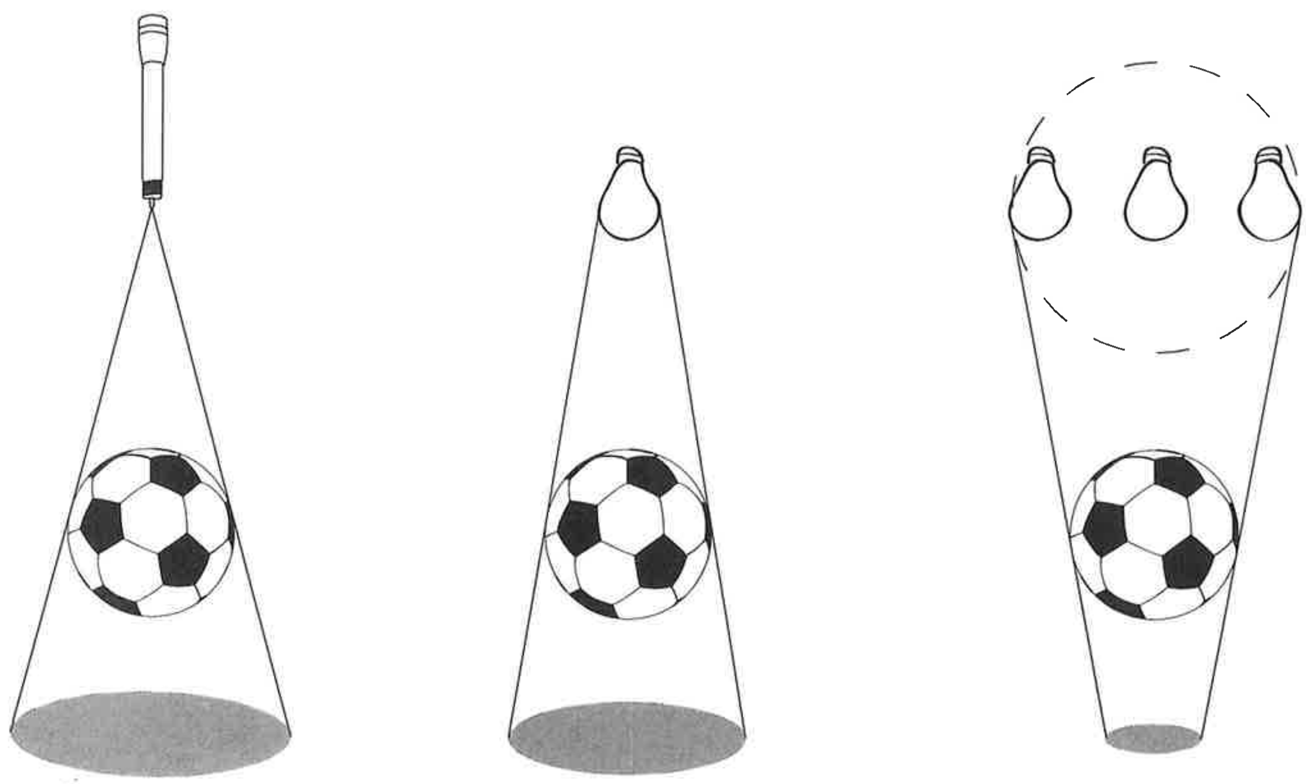


Figure 6.6: The angle of the umbral shadow cone depends on the size of the light source

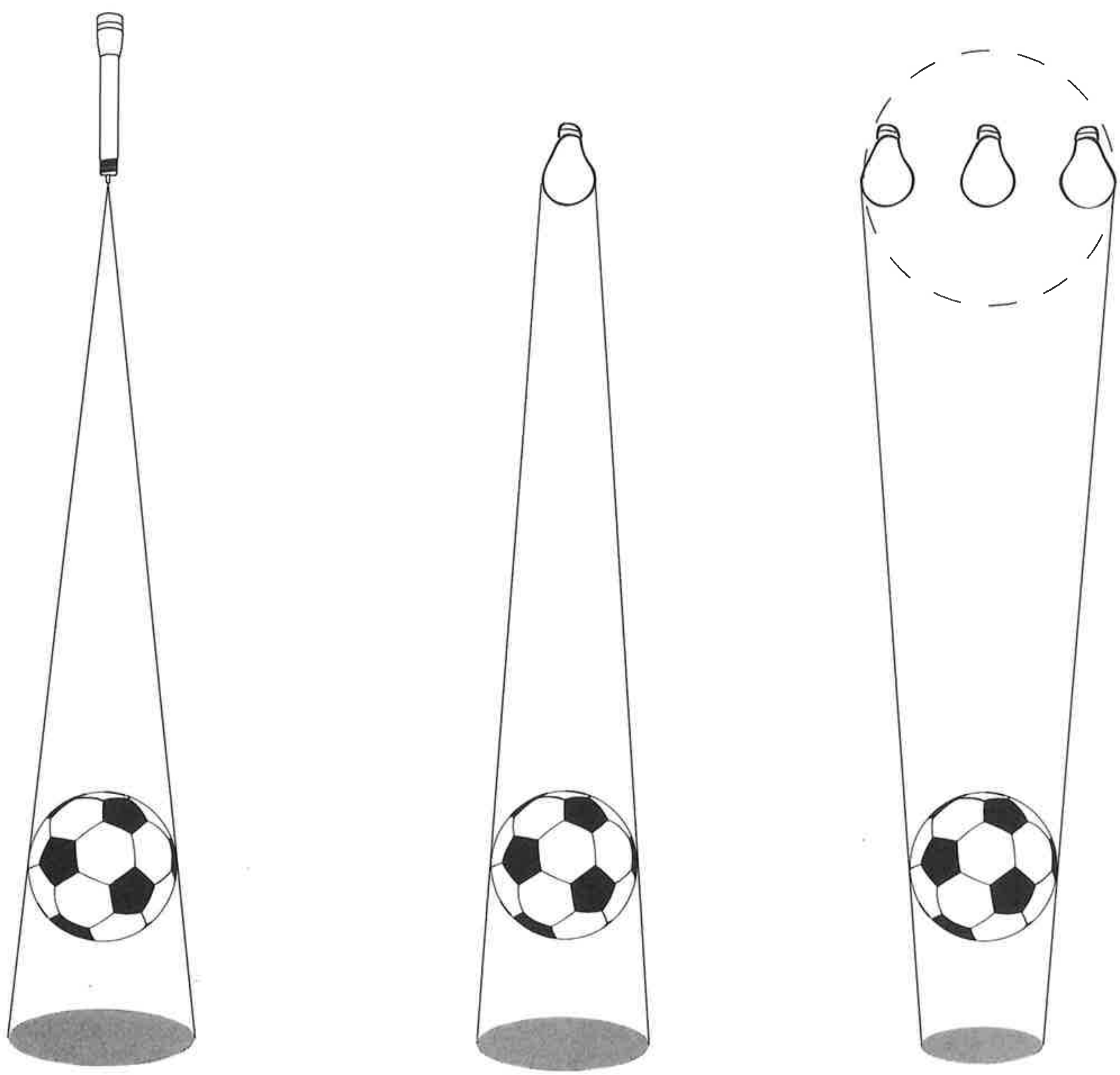


Figure 6.7: The angle of the umbral shadow cone also depends on the distance to the light source

We can calculate the angle of the cone for the point source if we know the size of the ball and the distance to the light:

$$\sin\theta = r / d$$

$$\theta = \arcsin(r / d)$$

Our standard soccer ball is 8¾ inches in diameter, which makes:

$$r = 4\tfrac{3}{8}$$

In Figure 6.8, the distance from the ball to the light is about 10 ball radii so:

$$d = 43\tfrac{3}{4}$$

Using these values in the equation above:

$$\sin\theta = 4\tfrac{3}{8} \div 43\tfrac{3}{4}$$

$$\theta = \arcsin(0.1)$$

$$\theta = 5\tfrac{3}{4}^\circ$$

The calculation is slightly different when the light source is larger, as shown in Figure 6.9. When the light is bigger than a point, but smaller than the ball, the angle is formed by the distance to the light (d) and the difference between the radius of the object (r) and the radius of the light (s):

$$\sin\theta = (r - s) / d$$

$$\theta = \arcsin[(r - s) / d]$$

When the light source is larger than the ball, as shown in Figure 6.10, the cone tapers in by the ratio of the difference between the light (s) and the ball (r) divided by the distance (d):

$$\sin\theta = (s - r) / d$$

$$\theta = \arcsin[(s - r) / d]$$

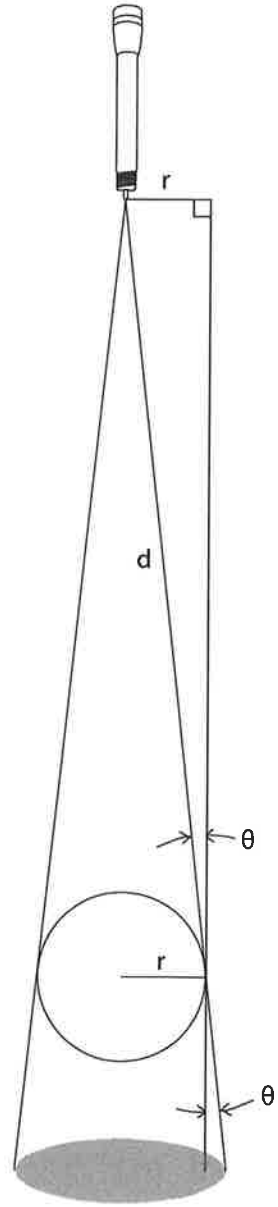


Figure 6.8: Calculating the angle of the shadow cone for a point source

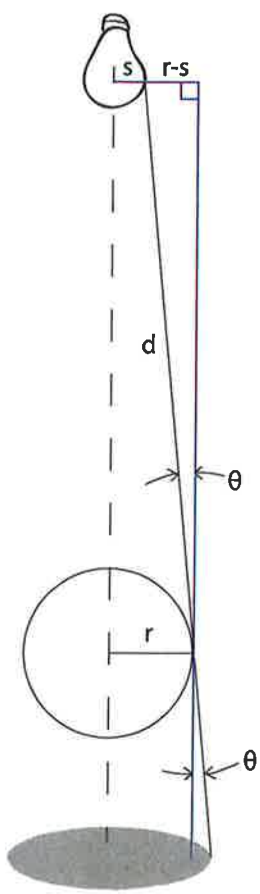


Figure 6.9: Calculating the angle of the shadow cone for a larger source

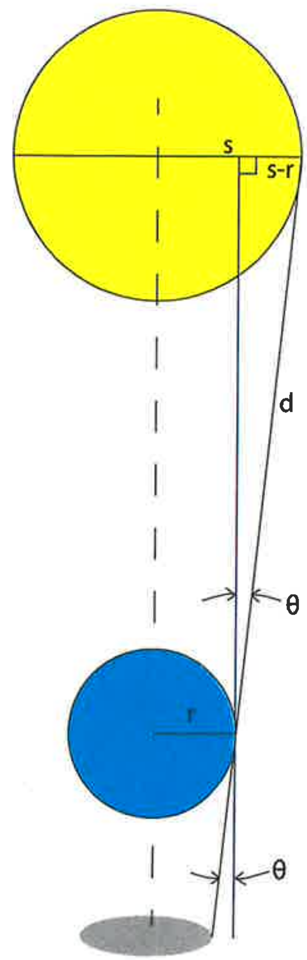


Figure 6.10: Calculating the angle of the shadow cone for a source larger than the object

Another way to think about how the shadow cone is formed is to imagine a straight cylinder extending from the ball to the light. Only the part of the light that hangs over the edge of the cylinder contributes to the taper of the umbral cone.

The ratio of the overhang to the distance determines the angle of the shadow cone. The taper of the shadow cone is equal to the apparent angle of the overhang. See Figure 6.11.

Now let's investigate the phenomenon of penumbra. When light comes from a point source, there is no penumbra. You either have shadow or you don't. See Figure 6.12

It's when the light source is extended that penumbra comes into play. Penumbra is the area of the shadow where part of the light shines, but part is blocked. If you looked at the light from the penumbral shadow, you would see a fraction of the light bulb. See Figure 6.13.

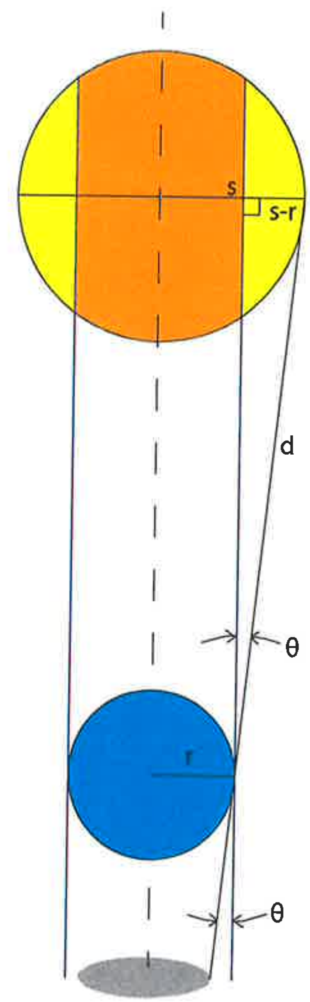


Figure 6.11: The angle of the shadow cone is the same as the apparent angle of the overhang

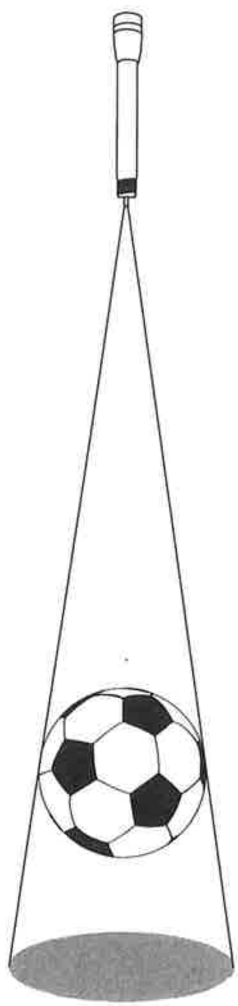


Figure 6.12: The shadow from a point light source does not have a penumbra

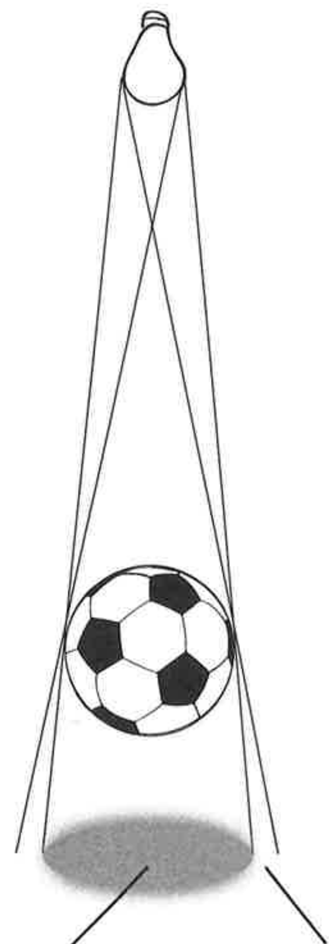
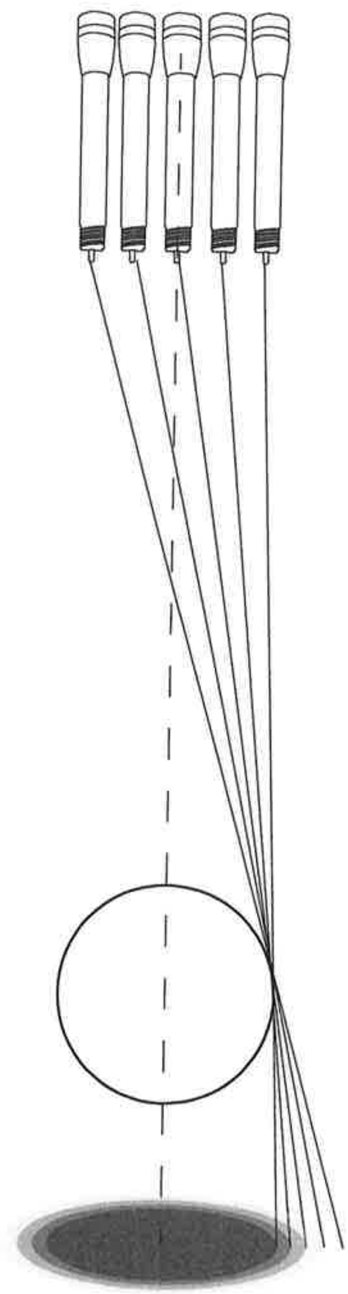


Figure 6.13: When the light source is extended, a penumbra is formed



Imagine that the large light source is made up of an array of point sources as in Figure 6.14. No light from any of the lights reaches the umbral area directly beneath the ball. As you move out, first one light source is visible, then as you move farther a second, and so on until all the lights are visible. The more light that reaches an area, the lighter the shadow. When none of the light is blocked, there is no shadow.

What is the angle of a penumbral shadow? Compare Figures 6.15 and 6.16. In both cases, the angle of the penumbra fringe is equal to the apparent angle of the light source.

Figure 6.17 shows that the size of the ball does not affect the penumbra. The umbra of Figure 6.17 is smaller than in Figure 6.16, but the penumbral fringe is the same angle. The angle of the penumbra depends only on the apparent angle of the light source, not on the size of the object.

Figure 6.14: The more light that reaches an area, the lighter the shadow.

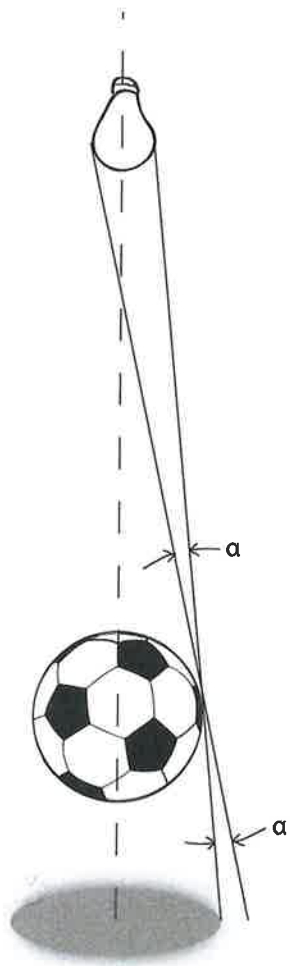


Figure 6.15: The angle of penumbra

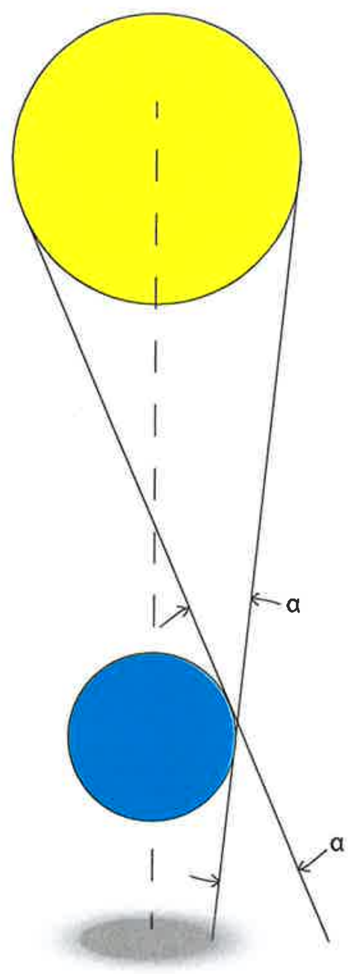


Figure 6.16: Angle of penumbra for a larger light source

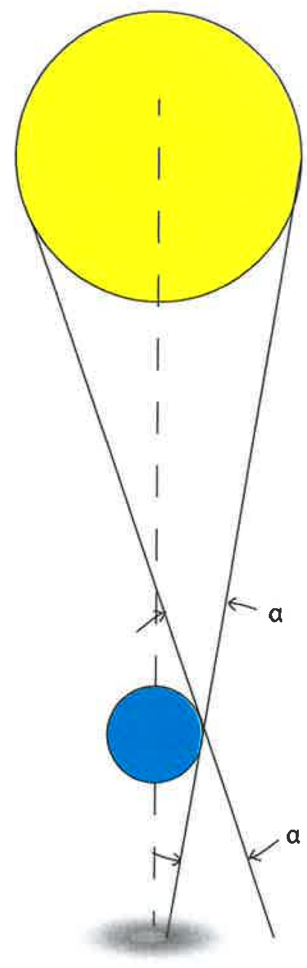


Figure 6.17: Angle of penumbra does not depend on object size

The apparent angle of the light source (or of any distant object) can be found by measuring the radius of the object (s) and the distance to the object (d):

$$\sin(\alpha / 2) = s / d$$

$$(\alpha / 2) = \arcsin(s / d)$$

$$\alpha = 2 \times \arcsin(s / d)$$

This is illustrated in Figure 6.18.

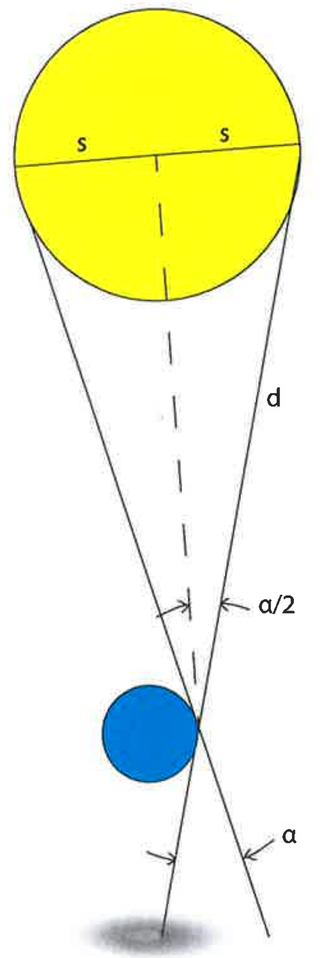


Figure 6.18: Measuring apparent angle to find penumbral angle

Let's use these equations to find the shadow angles cast by a soccer ball lighted by a standard light bulb. The soccer ball measures 8¾″ and the light bulb is 2⅜″ across. The ball is 35″ from the bulb.

Refer to Figure 6.19. To find the angle of the umbra, we find the overhang of the ball from the bulb by subtracting the radii:

$$\text{radius of the ball: } 8\tfrac{3}{4} \div 2 = 4\tfrac{3}{8}$$

$$\text{radius of the bulb: } 2\tfrac{3}{8} \div 2 = 1\tfrac{3}{16}$$

$$\text{difference: } 4\tfrac{3}{8} - 1\tfrac{3}{16} = 3\tfrac{3}{16}$$

$$\sin\theta = 3\tfrac{3}{16} \div 35$$

$$\theta \approx 5\tfrac{1}{4}°$$

To find the penumbral angle, as illustrated in Figure 6.20, find the apparent angle of the bulb:

$$\sin(\alpha / 2) = 1\tfrac{3}{16} \div 35$$

$$\alpha \approx 4°$$

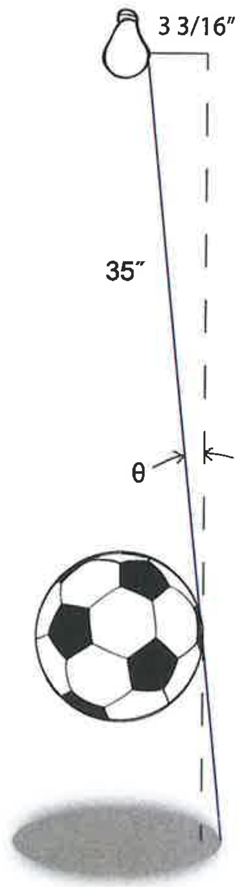


Figure 6.19: Finding umbral angle

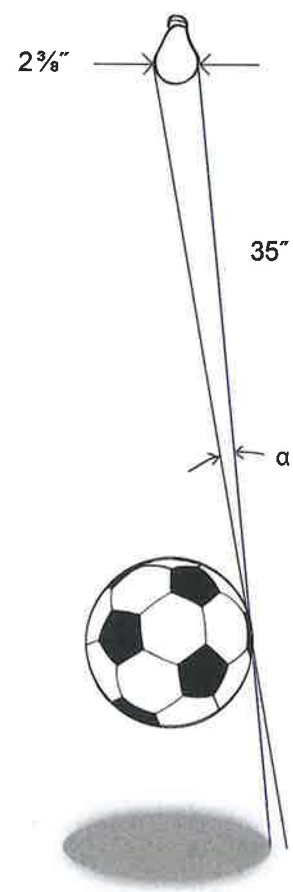


Figure 6.20: Finding penumbral angle

Now that we understand how shadows taper, let's return to the question asked at the opening of the chapter. Why does the soccer ball cast a smaller, fuzzy shadow in the sunlight?

According to Aristarchus, the Sun has a radius of at least 23,000 miles and is at least 5,000,000 miles away. What should the umbral and penumbral angles of the shadow be?

To find the umbra, first determine the difference in radius between the Sun and the ball:

$$23{,}000 \text{ miles} - 4\tfrac{3}{8} \text{ inches} \approx 23{,}000 \text{ miles}$$

$$\sin\theta = 23{,}000 \div 5{,}000{,}000$$

$$\theta \approx \tfrac{1}{4}^\circ$$

To find the penumbra, we start with the apparent angle of the Sun. We know that the distance to the Sun is 110 sun-diameters, or 220 sun-radii. (We could just use the distance in miles from above, but the 110 to 1 ratio is how we found those numbers in the first place.)

Apparent angle of Sun:

$$\sin(\alpha / 2) = 1 \div 220$$

$$\alpha \approx \tfrac{1}{2}^\circ$$

Now we have the angles of the umbra and penumbra. We can calculate the sizes of the umbra and penumbra 5 feet behind an 8¾″ soccer ball, as shown in Figures 6.21 and 6.22.

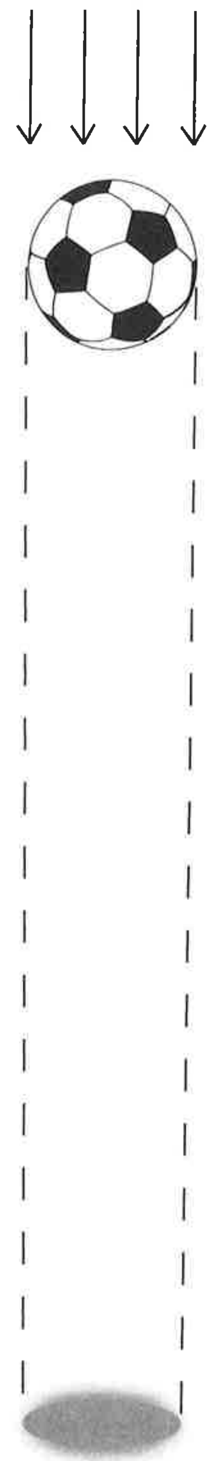


Figure 6.21: Shadow of a soccer ball in sunlight

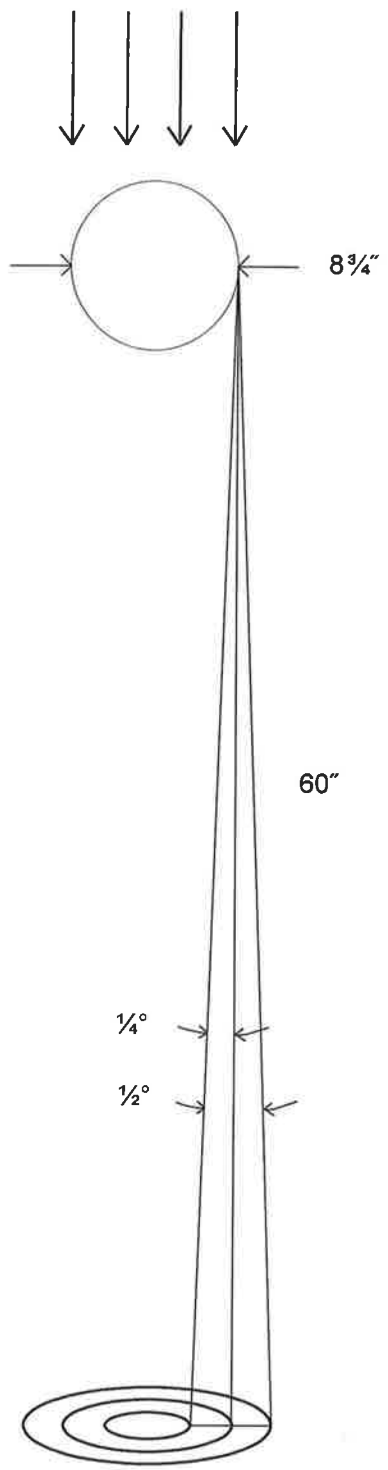


Figure 6.22: Size of the soccer balls shadow.

First we'll calculate the umbra. The umbra tapers in ¼° per side, so we can calculate how much smaller the radius of the umbra is (Δr_u) than the radius of the ball:

$$\sin ¼° = \Delta r_u \div 60''$$

$$\Delta r_u \approx ¼ \text{ inch}$$

The umbra tapers in by ¼″ on each side. This explains why we measured the umbra to be ½″ smaller than the ball, 8¼″ instead of 8¾″.

Next we'll calculate penumbra. The penumbra tapers out from the edge of the umbra by ½″ on each side. How much larger (Δr_p) than the umbra is the penumbra?

$$\sin ½° = \Delta r_p \div 60''$$

$$\Delta r_p \approx ½ \text{ inch}$$

The penumbra is ½″ larger than the umbra on each side,

$$8¼'' + ½'' + ½'' = 9¼''$$

We can't measure the edge of the penumbra because your eye can't see the difference between full sunlight and half sunlight. The penumbra just appears as a fuzzy fringe on the umbra.