

Measurement and Uncertainty

Kerry Kuehn

Wisconsin Lutheran College,
8800 West Bluemound Rd., Milwaukee,
WI 53226*

I. OVERVIEW

Working with experimental uncertainty often seems superfluous. When someone tells you that the tea is hot, you do not need them to specify that it is 192 ± 5 degrees Fahrenheit to know not to drink it quite yet. In scientific studies, however, in order to state whether one's results are consistent with another researcher's results, he or she must assign an experimental uncertainty to his or her measurements. Any measurement which does not include an error estimate is practically meaningless.

Consider the following example of assigning experimental uncertainty. A simple pendulum consists of a mass hanging from a string of length L . In order to measure the acceleration of gravity, g , one may perform the following steps.

1. Measure the period of time, T , for the pendulum to swing back and forth (one period of oscillation). T is a directly measured quantity.
2. Measure the length of the string L . L is also a directly measured quantity.
3. Use the formula

$$g = 4\pi^2 L/T^2 \quad (1)$$

to calculate the acceleration of gravity g , which is a derived quantity.

4. Determine the uncertainty in g by
 - (a) assigning uncertainties to T and L , and then
 - (b) propagating these uncertainties to find the uncertainty in g .

This chapter is written to address point number 4. Specifically, it will explain a general procedure of how to assign experimental uncertainties to directly measured quantities and then how to propagate these uncertainties to the derived quantities.

II. MEASUREMENT TERMINOLOGY

Let's begin by defining some terms that appear often when talking about scientific measurements.

- **uncertainty**: the degree of inexactness of experimental results
- **absolute uncertainty**: uncertainty expressed in same units as result (e.g. The absolute uncertainty in the length of the string may be 0.5 cm. We say that $\delta L = 0.5$ cm.)
- **relative uncertainty**: uncertainty expressed as a fraction of total result (e.g. The relative uncertainty in the length of the string may be 0.01, or 1%, if the absolute uncertainty, δL , is 0.5 cm, and $L = 50$ cm. We say that $\delta L/L = 0.01$.)
- **resolution**: smallest interval which can be meaningfully read from a device (e.g. A meter stick may have a resolution of 0.5 mm if it has ticks every millimeter)
- **rounding**: omitting digits beyond measurement resolution (e.g. We may round the measured length to the nearest half millimeter.)
- **precision**: the number of digits we can quote in our result (e.g. We might quote the length of a string as $L = 42.5$ cm, rather than 42.52 cm, if the resolution of the meter stick is only 0.5 cm.) The last recorded number is considered doubtful.
- **significant figures**: the number of trustworthy digits in a number. (e.g. The number $L = 42.5$ cm has three significant figures.)
- **systematic error**: mistake in the way the experiment was done (e.g. Perhaps the meter stick is warped by moisture and hence we underestimate any measurement we take with it.)
- **accuracy**: the degree to which our experiment is free of error (e.g. Our measurement may be inaccurate if our meter stick is bent.)
- **mean**: the average of N measured quantities

$$\bar{T} = \frac{1}{N} \sum_{i=1}^N T_i \quad (2)$$

* kerry.kuehn@wlc.edu; http://web.me.com/kerry_kuehn/Physics

- **standard deviation:** tells us about the average distance that any particular data point lies from the mean

$$\sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^N (T_i - \bar{T})^2} \quad (3)$$

If the measurements scatter according to what is called a "normal distribution", then the probability that any measurement, for example a measurement of the period of a single oscillation, T , lies within one standard deviation of the mean value, \bar{T} , is 67% or within two standard deviations of the mean value is 95%.

- **variance:** the square of the standard deviation:

$$var = \sigma^2 \quad (4)$$

III. COMPUTATIONS AND SIGNIFICANT FIGURES

The rightmost figure which is recorded in a result is considered to be doubtful. The following should be observed when manipulating such numbers.

- When adding or subtracting numbers, drop any figure in the sum to the right of the column containing the first doubtful figure. For example

$$104.4 + 40.01 = 144.4$$

- When multiplying or dividing numbers, keep one more significant figure than the smallest number of significant figures in the factors.

$$1.111 \times 7.7 = 8.55$$

IV. DIRECTLY MEASURED QUANTITIES

There are two ways of assigning experimental uncertainty to directly measured quantities.

1. **Estimation** The experimental uncertainty, σ , may be estimated from the resolution of the measuring apparatus. For example, the period of one oscillation of a pendulum as measured by a stopwatch might be $T = 1.325 \pm 0.005 \text{ cm}^1$.

¹ By measuring T N times, the uncertainty, $\bar{\sigma}$ is reduced to

$$\bar{\sigma} = \frac{\sigma}{\sqrt{N}}. \quad (5)$$

2. **Repeated measurement** Alternatively, the uncertainty, σ , may itself be obtained by repeated measurements. For example, the period of one oscillation of the pendulum might be measured five times. In this case, the uncertainty is not assigned first, based on the resolution of the stopwatch. Rather, the standard deviation, defined below, serves as the experimental uncertainty.

Example

As an example of determining uncertainty in a directly measured quantity using repeated measurement, consider five measurements of the period, T , of oscillation of a simple pendulum:

trial	T_i	$(T_i - \bar{T})$	$(T_i - \bar{T})^2$
1	1.325	0.004	0.000016
2	1.303	-0.018	0.000324
3	1.331	0.010	0.000100
4	1.328	0.007	0.000049
5	1.318	-0.003	0.000009

$$\bar{T} = 1.321\text{s}$$

$$\sigma = \sqrt{\frac{0.000498\text{s}}{4}} = 0.0112\text{s}$$

$$\bar{\sigma} = \frac{0.0112\text{s}}{\sqrt{5}} = 0.00501\text{s}$$

$$\bar{T} = 1.3210 \pm 0.005\text{s}$$

V. DERIVED QUANTITIES

Usually in an experiment, the quantity of interest is found by manipulating directly measured quantities according to some mathematical formula. How does one assign an uncertainty to these derived quantities? Shown in Tab.I are some straightforward rules for determining the uncertainty in derived quantities from uncertainties in directly measured quantities. For a more thorough explanation of the propagation of experimental uncertainty in derived quantities, see, for example, (?)

Examples

1. The diameter of a circle is given as $D = 40.1 \pm 0.2\text{cm}$. Find both its circumference, C , and the uncertainty in the circumference, δC .

$$C = \pi \times D = \pi \times 40.1\text{cm} = 126.0\text{cm}$$

operation	rule
multiplication by constant K $Q = Kx$	absolute uncertainty $\delta Q = K\delta x$ relative uncertainty $\frac{\delta Q}{ Q } = \frac{\delta x}{ x }$
raising to a power $Q = Kx^n$	relative uncertainty $\frac{\delta Q}{ Q } = n \frac{\delta x}{ x }$
addition or subtraction $Q = x \pm y \pm \dots$	absolute uncertainty $\delta Q = \sqrt{(\delta x)^2 + (\delta y)^2 + \dots}$
multiplication or division $Q = \frac{xy\dots}{u\dots}$	relative uncertainty $\frac{\delta Q}{ Q } = \sqrt{\left(\frac{\delta x}{x}\right)^2 + \left(\frac{\delta y}{y}\right)^2 + \left(\frac{\delta u}{u}\right)^2 + \dots}$
arbitrary function $Q = f(x)$	relative uncertainty $\delta Q = \left(\frac{ df }{ dx }\right) \delta x$

TABLE I Rules for uncertainty propagation.

The operation we use to compute the absolute uncertainty in circumference is *multiplication by a constant* with $K = \pi$.

$$\delta C = \pi \times \delta D = \pi(0.2\text{cm}) = 0.63\text{cm}$$

Since the circumference has only one significant figure to the right of the decimal, we can round this to

$$\delta C = 0.6\text{cm}.$$

We can also compute the relative uncertainty

$$\frac{\delta C}{C} = \frac{\delta D}{D} = \frac{0.2\text{cm}}{40.1\text{cm}} = 0.0050 = 0.50\%.$$

From the relative uncertainty we can compute the absolute uncertainty

$$\delta C = 0.0050 \times 126.0\text{cm} = 0.63 \simeq 0.6\text{cm}.$$

Notice that this gives the same result as our previous calculation of the absolute uncertainty.

- Find the area, and the uncertainty in the area, of the circle from the previous example.

$$A = \frac{\pi}{4} \times D^2 = \frac{\pi}{4} \times (40.1\text{cm})^2 = 1263\text{cm}^2$$

The operation we use to compute the relative uncertainty is *raising to a power* with $n = 2$.

$$\frac{\delta A}{A} = 2 \times \frac{\delta D}{D} = 2 \times \frac{0.2\text{cm}}{40.1\text{cm}} = 2 \times 0.010.$$

From the relative uncertainty in A we may compute the absolute uncertainty

$$\delta A = 0.020 \times 1263\text{cm}^2 = 25.26\text{cm}^2 \simeq 25\text{cm}^2.$$

- The length and width of a rectangle are given as $L = 85.0 \pm 0.2\text{cm}$ and $W = 29.5 \pm 0.2\text{cm}$. Find the perimeter P and the uncertainty δP .

$$P = 2(L + W) = 2(85.0\text{cm} + 29.5\text{cm}) = 229.0\text{cm}.$$

The operations used to compute the uncertainty in perimeter are *addition* and *multiplication by a constant*. First, we find the uncertainty in $(L + W)$ using the addition rule.

$$\delta(L+W) = \sqrt{(\delta L)^2 + (\delta W)^2} = \sqrt{(0.2\text{cm})^2 + (0.2\text{cm})^2} = 0.28\text{cm}.$$

Then we may find the uncertainty in the perimeter using the *multiplication by a constant* rule with $K = 2$.

$$\delta P = 2 \times \delta(L + W) = 2 \times 0.28\text{cm} = 0.56\text{cm} \simeq 0.6\text{cm}.$$

- Find the area, and the uncertainty in the area, of the rectangle from the previous example.

$$A = L \times W = 85.0\text{cm} \times 29.5\text{cm} = 2508\text{cm}^2.$$

The operation used to compute the relative uncertainty is *multiplication*.

$$\frac{\delta A}{A} = \sqrt{\left(\frac{\delta L}{L}\right)^2 + \left(\frac{\delta W}{W}\right)^2} = \sqrt{\left(\frac{0.2\text{cm}}{85.0\text{cm}}\right)^2 + \left(\frac{0.2\text{cm}}{29.5\text{cm}}\right)^2}$$

From the relative uncertainty in A we can compute the absolute uncertainty.

$$\delta A = 0.0072 \times 2508\text{cm}^2 = 18.0\text{cm}^2 \simeq 18\text{cm}^2.$$

5. Exercise: The length of a simple pendulum is given as $L = 42.5 \pm 0.5\text{cm}$. The period of oscillation is given as $T = 1.321 \pm 0.007\text{s}$. Find the acceleration of gravity, g , and the uncertainty, δg , using Eq.1. You will need to use the rule for *raising to a power*, then for *division*, then for *multiplication by a constant*.